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## RESEARCHES IN THE LUNAR THEORY.

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### CHAPTER II.

*Determination of the inequalities which depend only on the ratio of the mean motions of the sun and moon.*

If the path of a body, whose motion satisfies the differential equations

$$\frac{d^2x}{dt^2} - 2n' \frac{dy}{dt} + \left[ \frac{\mu}{r^3} - 3n'^2 \right] x = 0,$$

$$\frac{d^2y}{dt^2} + 2n' \frac{dx}{dt} + \frac{\mu}{r^3} y = 0,$$

intersect the axis of  $x$  at right angles, the circumstances of motion, before and after the intersection, are identical, but in reverse order with respect to the time. That is, if  $t$  be counted from the epoch when the body is on the axis of  $x$ , we shall have

$$x = \text{function } (t^2), \quad y = t \cdot \text{function } (t^2).$$

For if, in the differential equations, the signs of  $y$  and  $t$  are reversed, but that of  $x$  left unchanged, the equations are the same as at first.

A similar thing is true if the path intersect the axis of  $y$  at right angles ; for if the signs of  $x$  and  $t$  are reversed, while that of  $y$  is not altered, the equations undergo no change.

Now it is evident that the body may start from a given point on, and at right angles to, the axis of  $x$ , with different velocities ; and that, within certain limits, it may reach the axis of  $y$ , and cross the same at correspondingly different angles. If the right angle lie between some of these, we judge, from the principle of continuity, that there is some intermediate velocity with which the body would arrive at and cross the axis of  $y$  at right angles.

The difficulty of this question does not permit its being treated by a literal analysis; but the tracing of the path of the body, in numerous special cases, by the application of mechanical quadratures to the differential equations, enables us to state the following circumstances:—

If the body be projected at right angles to, and from a point on, the axis of  $x$ ; whose distance from the origin is less than  $0.33 \dots \sqrt[3]{\frac{\mu}{n^2}}$ , there is at least one (near the limit there are two) value of the initial velocity, with which the body, in arriving at the axis of  $y$ , will cross it at right angles. Beyond this limit it appears no initial velocity will serve to make the body reach the axis of  $y$  under the stated condition.

If the body move from one axis to the other and cross both of them perpendicularly, it is plain, from the preceding developments, that its orbit will be a closed curve symmetrical with respect to both axes. Thus is obtained a particular solution of the differential equations. While the general integrals involve four arbitrary constants, this solution, it is plain, has but two, which may be taken to be the distance from the origin at which the body crosses the axis of  $x$  and the time of crossing.

Certain considerations, connected with the employment of Fourier's Theorem and the possibility of developing functions in infinite series of periodic terms, show that, in this solution, the co-ordinates of the body can be represented, in a convergent manner, by series of the following form,

$$x = A_0 \cos [\nu (t - t_0)] + A_1 \cos 3 [\nu (t - t_0)] + A_2 \cos 5 [\nu (t - t_0)] + \dots, \\ y = B_0 \sin [\nu (t - t_0)] + B_1 \sin 3 [\nu (t - t_0)] + B_2 \sin 5 [\nu (t - t_0)] + \dots,$$

where  $t_0$  denotes the time the body crosses the axis of  $x$ , and  $\frac{2\pi}{\nu}$  is the time of a complete revolution of the body about the origin. We may regard  $\nu$  and  $t_0$  as the arbitrary constants introduced by integration; the coefficients  $A_0, A_1 \dots B_1, B_2 \dots$  are functions of  $\mu, n'$  and  $\nu$ .

For convenience sake we may put

$$A_i = a_i + a_{-i-1}, \quad B_i = a_i - a_{-i-1}.$$

Then,  $\tau$  being put for  $\nu (t - t_0)$ , the series, given above, may be written

$$x = \sum_i a_i \cos (2i + 1) \tau,$$

$$y = \sum_i a_i \sin (2i + 1) \tau,$$

the summation being extended to all integral values positive and negative, zero included, for  $i$ . By adopting polar co-ordinates such that

$$x = r \cos \phi, \quad y = r \sin \phi,$$

and writing  $v$  for  $\phi - \tau$ , that is, for the excess of the true over the mean longitude of the moon, the last equations are equivalent to

$$r \cos v = \sum_i a_i \cos 2i\tau,$$

$$r \sin v = \sum_i a_i \sin 2i\tau.$$

In order to avoid the multiplication of series of sines and cosines, and reduce everything to an algebraic form, for  $x$  and  $y$ , we substitute the imaginary variables  $u$  and  $s$ , and put  $\zeta = e^{\nu \sqrt{-1}}$ . We have then

$$u = \sum_i a_i \zeta^{2i+1}, \quad s = \sum_i a_{-i-1} \zeta^{2i+1}.$$

$\zeta$  will always be employed as the independent variable in place of  $t$  or  $\tau$ .

Denoting the operation  $\zeta \frac{d}{d\zeta} = -\sqrt{-1} \frac{d}{d\tau}$  by the symbol  $D$ , so that, in general,

$$D(a\zeta^i) = ia\zeta^i,$$

and taking the liberty of separating this symbol as if it were a multiplier, and moreover putting

$$m = \frac{n'}{\nu} = \frac{n'}{n - n'}, \quad x = \frac{\mu}{\nu^2},$$

the differential equations, determining  $u$  and  $s$ , given in the preceding chapter, may be written

$$\left[ D^2 + 2mD + \frac{3}{2}m^2 - \frac{x}{(us)^{\frac{3}{2}}} \right] u + \frac{3}{2}m^2 s = 0,$$

$$\left[ D^2 - 2mD + \frac{3}{2}m^2 - \frac{x}{(us)^{\frac{3}{2}}} \right] s + \frac{3}{2}m^2 u = 0.$$

It will be noticed that either of these equations can be derived from the other by interchanging  $u$  and  $s$  and reversing the sign of  $m$  or  $D$ . We may also remind the reader that they determine rigorously all the parts of the lunar coordinates which depend only on the ratio of the mean motions of the sun and moon and on the lunar eccentricity. The Jacobian integral, in the present notation, is

$$Du \cdot Ds + \frac{2x}{(us)^{\frac{1}{2}}} + \frac{3}{4} m^2 (u + s)^2 = C.$$

The most ready method of getting the values of the coefficients  $a_i$ , is that of undetermined coefficients; the values of  $u$  and  $s$ , expressed by the preceding summations with reference to  $i$ , being substituted in the differential equations, the resulting coefficient of each power of  $\zeta$ , in the left members, is equated to zero, which furnishes a series of equations of condition sufficient

to determine all the quantities  $a_i$ . For this purpose we may evidently employ any two independent combinations of the three equations last written, and it will be advisable to form these combinations in such a manner that the process of deriving the equations of condition may be facilitated in the largest degree. Now it will be recognized that the presence of the term

$\frac{\varkappa}{(us)^{\frac{3}{2}}}$ , in one of the factors of the differential equations, is a hindrance to their ready integration, being the single thing which prevents them from being linear with constant coefficients. Hence we avail ourselves of the possibility of eliminating it. Multiplying the first differential equation by  $s$ , and the second by  $u$ , and taking, in succession, the sum and difference,

$$\begin{aligned}uD^2s + sD^2u - 2m(uDs - sDu) - \frac{2\varkappa}{(us)^{\frac{1}{2}}} + \frac{3}{2}m^2(u + s)^2 &= 0, \\uD^2s - sD^2u - 2m(uDs + sDu) + \frac{3}{2}m^2(u^2 - s^2) &= 0,\end{aligned}$$

then, adding to the first of these the integral equation, and retaining the second as it is, we have, as the final differential equations to be employed,

$$\begin{aligned}D^2(us) - Du \cdot Ds - 2m(uDs - sDu) + \frac{9}{4}m^2(u + s)^2 &= C, \\D(uDs - sDu - 2mus) + \frac{3}{2}m^2(u^2 - s^2) &= 0.\end{aligned}$$

It must be pointed out, however, that these equations are not, in all respects, a complete substitute for the original equations. It will be seen that  $\mu$  or  $\varkappa$ , an essential element in the problem, has disappeared from them, and that, in integration, an arbitrary constant, in excess of those admissible, will present itself. This will be eliminated by substituting the integrals found in one of the original differential equations, in which  $\mu$  or  $\varkappa$  is present; the result being an equation of condition by which the superfluous constant can be expressed in terms of  $\mu$  and the remaining constants.

We remark that the left members of our differential equations are homogeneous and of two dimensions with respect to  $u$  and  $s$ . If the first were differentiated, the constant  $C$  would disappear, and both equations would be homogeneous in all their terms. This property renders them exceedingly useful when equations of condition are to be obtained between the coefficients of the different periodic terms of the lunar co-ordinates, and it is for this purpose that we have given them their present form.

From the signification of the symbol  $D$ ,

$$Du = \sum_i . (2i + 1) a_i \zeta^{2i+1}, \quad Ds = \sum_i . (2i + 1) a_{-i-1} \zeta^{2i+1},$$

$$D^2u = \sum_i . (2i + 1)^2 a_i \zeta^{2i+1}, \quad D^2s = \sum_i . (2i + 1)^2 a_{-i-1} \zeta^{2i+1};$$

also

$$us = \sum_j . [\sum_i . a_i a_{i-j}] \zeta^{2j},$$

$$u^2 = \sum_j . [\sum_i . a_i a_{-i+j-1}] \zeta^{2j},$$

$$s^2 = \sum_j . [\sum_i . a_i a_{-i-j-1}] \zeta^{2j},$$

$$Du \cdot Ds = - \sum_j . [\sum_i . (2i + 1)(2i - 2j + 1) a_i a_{i-j}] \zeta^{2j},$$

$$uDs - sDu = - 2\sum_j . [\sum_i . (2i - j + 1) a_i a_{i-j}] \zeta^{2j},$$

where the summations with reference to  $j$  have the same extension as those with reference to  $i$ . On substituting these expressions in the differential equations, and equating the general coefficients of  $\zeta^{2j}$  to zero, we get

$$\sum_i . \left[ (2i + 1)(2i - 2j + 1) + 4j^2 + 4(2i - j + 1) m + \frac{9}{2} m^2 \right] a_i a_{i-j} + \frac{9}{4} m^2 \sum_i . \left[ a_i a_{-i+j-1} + a_i a_{-i-j-1} \right] = 0,$$

$$4j \sum_i . \left[ 2i - j + 1 + m \right] a_i a_{i-j} - \frac{3}{2} m^2 \sum_i . \left[ a_i a_{-i+j-1} - a_i a_{-i-j-1} \right] = 0,$$

which hold for all integral values of  $j$  both positive and negative except that, when  $j = 0$ , the right member of the first equation is  $C$  instead of 0; but as the second equation is an identity for  $j = 0$ , for the present this value of  $j$  will be excluded from consideration.

By multiplying the first equation by 2, and the second by 3, and taking in succession the difference and sum, the simpler forms are obtained,

$$\sum_i . [8i^2 - 8(4j-1)i + 20j^2 - 16j + 2 + 4(4i - 5j + 2) m + 9m^2] a_i a_{i-j} + 9m^2 \sum_i . a_i a_{-i+j-1} = 0,$$

$$\sum_i . [8i^2 + 8(2j+1)i - 4j^2 + 8j + 2 + 4(4i + j + 2) m + 9m^2] a_i a_{i-j} + 9m^2 \sum_i . a_i a_{-i-j-1} = 0.$$

These two equations are not distinct from each other, when negative, as well as positive values, are attributed to  $j$ . For if, in the expression under the first sign of summation in the first equation, we substitute, which is allowable, for  $i$ ,  $i-j$ , and  $-j$  for  $j$  throughout the equation, the result is identical with the second equation. This is explained by the fact that we get all the independent equations of condition, these equations are capable of furnishing, by attributing only positive values to  $j$ . Hence, allowing  $j$  to receive positive

and negative values, all the equations of condition can be represented by a unique formula.

Although the number of these equations is infinite, and also that of the coefficients  $a_i$ , it is not difficult to see that the first ought to be regarded as one less than the second; and that, in consequence of the bi-dimensional character of the equations, they suffice to determine the ratio of any two of the quantities  $a_i$  in terms of  $m$ . It will be seen, from developments to be given shortly, that if  $m$  is regarded as a small quantity of the first order,  $a_i$  is of the  $\pm 2i^{\text{th}}$  order. It will be advisable then to select  $a_0$  as the coefficient to which to refer all the rest; and we shall have, in general,

$$a_i = a_0 F(m).$$

The equations of condition, as written above, determine the  $a_i$  in pairs; that is, if we put  $j = 1$ , we have the equations suitable for determining  $a_1$  and  $a_{-1}$ , and, in general, the equations, as written, determine  $a_j$  and  $a_{-j}$ . And, as they involve both these quantities, it will be advantageous to eliminate approximately each in succession, as far as that can be done without depriving the equations of their bi-dimensional character.

By putting, in succession, in the terms under the first sign of summation,  $i = 0$  and  $i = j$ , it will be found that these equations contain, severally, the terms

$$\begin{aligned} [20j^2 - 16j + 2 - 4(5j - 2)m + 9m^2] a_0 a_{-j} \\ + [-4j^2 - 8j + 2 - 4(j - 2)m + 9m^2] a_0 a_j, \\ [-4j^2 + 8j + 2 + 4(j + 2)m + 9m^2] a_0 a_{-j} \\ + [20j^2 + 16j + 2 + 4(5j + 2)m + 9m^2] a_0 a_j, \end{aligned}$$

which are the terms of principal moment in determining  $a_{-j}$  and  $a_j$ . Let us then multiply the first equation by

$$-4j^2 + 8j + 2 + 4(j + 2)m + 9m^2,$$

and the second by

$$-20j^2 + 16j - 2 + 4(5j - 2)m - 9m^2,$$

and, adding the products, divide the whole by

$$48j^2 [2(4j^2 - 1) - 4m + m^2].$$

Then, adopting the notation

$$\begin{aligned} [j, i] &= -\frac{i}{j} \frac{4(j-1)i + 4j^2 + 4j - 2 - 4(i-j+1)m + m^2}{2(4j^2 - 1) - 4m + m^2}, \\ [j] &= -\frac{3m^2}{16j^2} \frac{4j^2 - 8j - 2 - 4(j+2)m - 9m^2}{2(4j^2 - 1) - 4m + m^2}, \end{aligned}$$

$$(j) = -\frac{3m^2}{16j^2} \frac{20j^2 - 16j + 2 - 4(5j - 2)m + 9m^2}{2(4j^2 - 1) - 4m + m^2},$$

the system of equations, which determines the coefficients  $a_i$ , is represented by the unique formula

$$\Sigma_i \cdot [j, i] a_i a_{i-j} + [j] a_i a_{-i+j-1} + (j) a_i a_{-i-j-1} = 0,$$

where  $j$  must receive negative as well as positive values. It will be perceived that

$$[j, 0] = 0, \quad [j, j] = -1;$$

hence the last equation is in a form suitable for determining the value of  $a_j$ . The quantities  $[j, i]$ ,  $[j]$  and  $(j)$  admit of being expressed in a simpler manner; thus

$$[j, i] = -\frac{i}{j} + \frac{4i(j-i)}{j} \frac{j-1-m}{2(4j^2-1)-4m+m^2},$$

whence

$$[j, i] + [-j, -i] = -\frac{2i}{j} + \frac{8i(j-i)}{2(4j^2-1)-4m+m^2},$$

$$[j, i] - [-j, -i] = \frac{8i(i-j)}{j} \frac{1+m}{2(4j^2-1)-4m+m^2};$$

in addition

$$[j] = \frac{27}{16j^2} m^2 - \frac{3}{4j^2} \frac{19j^2 - 2j - 5 - (j+11)m}{2(4j^2-1)-4m+m^2} m^2,$$

$$(j) = -\frac{27}{16j^2} m^2 + \frac{3}{4j^2} \frac{13j^2 + 4j - 5 + (5j-11)m}{2(4j^2-1)-4m+m^2} m^2,$$

$$[j] + (-j) = -\frac{3}{2j} \frac{3j+1+2m}{2(4j^2-1)-4m+m^2} m^2,$$

$$[j] - (-j) = \frac{27}{8j^2} m^2 - \frac{3}{2j^2} \frac{16j^2 - 3j - 5 - (3j+11)m}{2(4j^2-1)-4m+m^2} m^2.$$

In making a first approximation to the values of the coefficients, one of the terms of the equation may be omitted; for, when  $j$  is positive, the term  $\Sigma_i (j) a_i a_{-i-j-1}$  is a quantity four orders higher than that of the terms of the lowest order contained in the equation; and, when  $j$  is negative, the same thing is true of  $\Sigma_i [j] a_i a_{-i+j-1}$ . Hence, with this limitation, the equation may be written in the two forms

$$\Sigma_i \cdot [j, i] a_i a_{i-j} + [j] a_i a_{-i+j-1} = 0,$$

$$\Sigma_i \cdot [-j, i] a_i a_{i+j} + (-j) a_i a_{-i+j-1} = 0.$$

where  $j$  takes only positive values.

From these two equations, by omitting all terms but those of the lowest order, we derive the following series of equations, determining the coefficients to the first degree of approximation,

$$a_0 a_1 = [1] a_0 a_0,$$

$$a_0 a_{-1} = (-1) a_0 a_0,$$

$$a_0 a_2 = [2] [a_0 a_1 + a_1 a_0] + [2, 1] a_1 a_{-1},$$

$$a_0 a_{-2} = (-2) [a_0 a_1 + a_1 a_0] + [-2, -1] a_1 a_{-1},$$

$$a_0 a_3 = [3] [a_0 a_2 + a_1 a_1 + a_2 a_0] + [3, 1] a_1 a_{-2} + [3, 2] a_2 a_{-1},$$

$$a_0 a_{-3} = (-3) [a_0 a_2 + a_1 a_1 + a_2 a_0] + [-3, -1] a_{-1} a_2 + [-3, -2] a_{-2} a_1,$$

$$a_0 a_4 = [4] [a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0] + [4, 1] a_1 a_{-3} + [4, 2] a_2 a_{-2} + [4, 3] a_3 a_{-1},$$

$$a_0 a_{-4} = (-4) [a_0 a_3 + a_1 a_2 + a_2 a_1 + a_3 a_0] + [-4, -1] a_{-1} a_3 + [-4, -2] a_{-2} a_2 \\ + [-4, -3] a_{-3} a_1,$$

.....

The law of these equations is quite apparent, and they can easily be extended as far as desired. The first two give the values of  $a_1$  and  $a_{-1}$ , the following two the values of  $a_2$  and  $a_{-2}$  by means of the values of  $a_1$  and  $a_{-1}$  already obtained, and so on, every two equations of the series giving the values of two coefficients by means of the values of all those which precede in the order of enumeration. A glance at the composition of these equations must convince us that all attempts to write explicitly, even this approximate value of  $a_i$ , would be unsuccessful on account of the excessive multiplicity of the terms. However, they may be regarded, in some sense, as giving the law of this approximate solution, since they exhibit clearly the mode in which each coefficient depends on all those which precede it. As to the degree of approximation afforded by these equations, when the values are expanded in series of ascending powers of  $m$ , the first four terms are obtained correctly in the case of each coefficient. Thus  $a_1$  and  $a_{-1}$  are affected with errors of the 6th order,  $a_2$  and  $a_{-2}$  with errors of the 8th order,  $a_3$  and  $a_{-3}$  with errors of the 10th order, and so on.

The values of these quantities can be determined either in the literal form, where the parameter  $m$  is left indeterminate, as has been done by Plana and Delaunay, or as numbers, which mode has been followed by all the earlier lunar-theorists and Hansen. In the latter case, one will begin by computing the numerical values of the quantities  $[j, i]$ ,  $[j]$  and  $(j)$ , corresponding to the assumed value of  $m$ , for all necessary values of the integers  $i$  and  $j$ .

The great advantage of our equations consists in this, that we are able to extend the approximation as far as we wish, simply by writing explicitly the terms, our symbols giving the law of the coefficients. How rapid is the approximation in the terms of these equations will be apparent, when we say, that, after a certain number of terms are written, in order to carry this four orders higher, it is necessary to add to each of them only four new terms; and, thereafter, every addition of four terms enables us to carry the approximation four orders farther.

The process which may be followed to obtain the values of the  $a_i$  with any desired degree of accuracy, is this:—the first approximate values will be got from the preceding group of equations until the  $a_i$  become of orders intended to be neglected; then one will recommence at the beginning, using the equations each augmented by the terms necessary to carry the approximation four orders higher; substituting in the new terms the values obtained from the first approximation, and, in the old, ascertaining what changes are produced by employing the more exact values instead of the first approximations. A second return to the beginning of the work will, in like manner, push the degree of exactitude four orders higher. In this way any required degree of approximation may be attained.

Whatever advantage the present process may have over those previously employed is plainly due to the use of the indeterminate integers  $i$  and  $j$ , which, although much used in the planetary theories, no one seems to have thought of introducing into the lunar theory. This enables us to perform a large mass of operations once for all.

For the purpose of making evident the preceding assertions, and because we shall have occasion to use them, we write below the equations determining the coefficients  $a_i$  correct to quantities of the 13th order inclusive.

$$\begin{aligned}
 a_0a_1 &= [1] [a_0^2 + 2a_{-1}a_1 + 2a_{-2}a_2] + (1) [a_{-1}^2 + 2a_0a_{-2} + 2a_1a_{-3}] \\
 &\quad + [1, -2] a_{-2}a_{-3} + [1, -1] a_{-1}a_{-2} + [1, 2] a_2a_1 + [1, 3] a_3a_2, \\
 a_0a_{-1} &= [-1] [a_{-1}^2 + 2a_0a_{-2} + 2a_1a_{-3}] + (-1) [a_0^2 + 2a_{-1}a_1 + 2a_{-2}a_2] \\
 &\quad + [-1, -3] a_{-3}a_{-2} + [-1, -2] a_{-2}a_{-1} + [-1, 1] a_1a_2 + [-1, 2] a_2a_3, \\
 a_0a_2 &= [2] [2a_0a_1 + 2a_{-1}a_2 + 2a_{-2}a_3] + (2) [2a_{-1}a_{-2} + 2a_0a_{-3} + 2a_1a_{-4}] \\
 &\quad + [2, -2] a_{-2}a_{-4} + [2, -1] a_{-1}a_{-3} + [2, 1] a_1a_{-1} + [2, 3] a_3a_1 + [2, 4] a_4a_2, \\
 a_0a_{-2} &= [-2] [2a_{-1}a_{-2} + 2a_0a_{-3} + 2a_1a_{-4}] + (-2) [2a_0a_1 + 2a_{-1}a_2 + 2a_{-2}a_3] \\
 &\quad + [-2, -4] a_{-4}a_{-2} + [-2, -3] a_{-3}a_{-1} + [-2, -1] a_{-1}a_1 + [-2, 1] a_1a_3 \\
 &\quad + [-2, 2] a_2a_4,
 \end{aligned}$$

$$\begin{aligned}
a_0 a_3 &= [3] [a_1^2 + 2a_0 a_2 + 2a_{-1} a_3] + (3) [a_{-2}^2 + 2a_{-1} a_{-3} + 2a_0 a_{-4}] \\
&\quad + [3, -1] a_{-1} a_{-4} + [3, 1] a_1 a_{-2} + [3, 2] a_2 a_{-1} + [3, 4] a_4 a_1, \\
a_0 a_{-3} &= [-3] [a_{-2}^2 + 2a_{-1} a_{-3} + 2a_0 a_{-4}] + (-3) [a_1^2 + 2a_0 a_2 + 2a_{-1} a_3] \\
&\quad + [-3, -4] a_{-4} a_{-1} + [-3, -2] a_{-2} a_1 + [-3, -1] a_{-1} a_2 + [-3, 1] a_1 a_4, \\
a_0 a_4 &= [4] [2a_1 a_2 + 2a_0 a_3 + 2a_{-1} a_4] + (4) [2a_{-2} a_{-3} + 2a_{-1} a_{-4} + 2a_0 a_{-5}] \\
&\quad + [4, -1] a_{-1} a_{-5} + [4, 1] a_1 a_{-3} + [4, 2] a_2 a_{-2} + [4, 3] a_3 a_{-1} + [4, 5] a_5 a_1, \\
a_0 a_{-4} &= [-4] [2a_{-2} a_{-3} + 2a_{-1} a_{-4} + 2a_0 a_{-5}] + (-4) [2a_1 a_2 + 2a_0 a_3 + 2a_{-1} a_4] \\
&\quad + [-4, -5] a_{-5} a_{-1} + [-4, -3] a_{-3} a_1 + [-4, -2] a_{-2} a_2 + [-4, -1] a_{-1} a_3 \\
&\quad + [-4, 1] a_1 a_5, \\
a_0 a_5 &= [5] [a_2^2 + 2a_1 a_3 + 2a_0 a_4] \\
&\quad + [5, 1] a_1 a_{-4} + [5, 2] a_2 a_{-3} + [5, 3] a_3 a_{-2} + [5, 4] a_4 a_{-1}, \\
a_0 a_{-5} &= (-5) [a_2^2 + 2a_1 a_3 + 2a_0 a_4] \\
&\quad + [-5, -4] a_{-4} a_1 + [-5, -3] a_{-3} a_2 + [-5, -2] a_{-2} a_3 + [-5, -1] a_{-1} a_4, \\
a_0 a_6 &= [6] [2a_2 a_3 + 2a_1 a_4 + 2a_0 a_5] \\
&\quad + [6, 1] a_1 a_{-5} + [6, 2] a_2 a_{-4} + [6, 3] a_3 a_{-3} + [6, 4] a_4 a_{-2} + [6, 5] a_5 a_{-1}, \\
a_0 a_{-6} &= (-6) [2a_2 a_3 + 2a_1 a_4 + 2a_0 a_5] \\
&\quad + [-6, -5] a_{-5} a_1 + [-6, -4] a_{-4} a_2 + [-6, -3] a_{-3} a_3 + [-6, -2] a_{-2} a_4 \\
&\quad + [-6, -1] a_{-1} a_5.
\end{aligned}$$

In the first approximation

$$\begin{aligned}
a_1 &= [1] a_0, \\
a_{-1} &= (-1) a_0, \\
a_2 &= [1] [2(2) + [2, 1] (-1)] a_0, \\
a_{-2} &= [1] [2(-2) + [-2, -1] (-1)] a_0,
\end{aligned}$$

or, explicitly in terms of  $m$ ,

$$\begin{aligned}
a_1 &= \frac{3}{16} \frac{6 + 12m + 9m^2}{6 - 4m + m^2} m^2 a_0, \\
a_{-1} &= -\frac{3}{16} \frac{38 + 28m + 9m^2}{6 - 4m + m^2} m^2 a_0,
\end{aligned}$$

and, after some reductions,

$$\begin{aligned}
a_2 &= \frac{27}{256} \frac{2 + 4m + 3m^2}{[6 - 4m + m^2] [30 - 4m + m^2]} \left[ 238 + 40m + 9m^2 - 32 \frac{29 - 35m}{6 - 4m + m^2} \right] m^4 a_0, \\
a_{-2} &= \frac{27}{64} \frac{2 + 4m + 3m^2}{[6 - 4m + m^2] [30 - 4m + m^2]} \left[ -28 - 7m + 24 \frac{7 - m}{6 - 4m + m^2} \right] m^4 a_0.
\end{aligned}$$

It is evident that, however far the approximation may be carried, the only quantities, involved as divisors in the values of the  $a_i$ , are the trinomials, whose general expression is

$$2(4j^2 - 1) - 4m + m^2,$$

or, particularizing, the series of divisors is

$$\begin{aligned} & 6 - 4m + m^2, \\ & 30 - 4m + m^2, \\ & 70 - 4m + m^2, \\ & \dots \dots \dots \end{aligned}$$

It will be remarked that they differ only in their first terms, which are independent of  $m$ . Hence any expression, involving several divisors, can always be separated into several parts, each involving only one divisor, without any actual division by a trinomial in  $m$ . For instance,

$$\begin{aligned} \frac{1}{[6 - 4m + m^2][30 - 4m + m^2]} &= \frac{1}{24} \frac{1}{6 - 4m + m^2} - \frac{1}{24} \frac{1}{30 - 4m + m^2}, \\ \frac{1}{[6 - 4m + m^2]^2[30 - 4m + m^2]} &= \frac{1}{24} \frac{1}{[6 - 4m + m^2]^2} - \frac{1}{24^2} \frac{1}{6 - 4m + m^2} \\ &+ \frac{1}{24^2} \frac{1}{30 - 4m + m^2}. \end{aligned}$$

Moreover when, after this transformation, any numerator contains more or other powers of  $m$  than two consecutive powers, it is clear it may be reduced so as to contain only these by eliminating the higher powers through subtracting certain multiples of the divisor which appears in the denominator, or, in other words, the fraction may be treated as if it were improper.

From this we gather that the value of  $a_i$  can be expressed thus

$$\begin{aligned} \frac{a_i}{a_0} &= M_0 + \frac{M_1}{6 - 4m + m^2} + \frac{M_2}{[6 - 4m + m^2]^2} + \frac{M_3}{[6 - 4m + m^2]^3} + \dots \\ &+ \frac{N_1}{30 - 4m + m^2} + \frac{N_2}{[30 - 4m + m^2]^2} + \frac{N_3}{[30 - 4m + m^2]^3} + \dots \\ &+ \frac{P_1}{70 - 4m + m^2} + \frac{P_2}{[70 - 4m + m^2]^2} + \frac{P_3}{[70 - 4m + m^2]^3} + \dots \\ &+ \dots \dots \dots \dots \dots \end{aligned}$$

where  $M_0, M_1 \dots N_1, N_2 \dots P_1, P_2 \dots$  are entire functions of  $m$  each of the form

$$Am^k + Bm^{k+1}.$$

The advantage of this method of treatment consists in that nothing, which is given by the successive approximations, would be lost, as must be the case

when the values are expanded in series of ascending powers of  $m$ . The preceding expressions, when put into this form, become

$$\frac{a_1 + a_{-1}}{a_0} = -3 \frac{2 + m}{6 - 4m + m^2} m^2,$$

$$\frac{a_1 - a_{-1}}{a_0} = 3 \left[ \frac{9}{8} - \frac{4 - 7m}{6 - 4m + m^2} \right] m^2,$$

$$\frac{a_2 + a_{-2}}{a_0} = \frac{3}{16} \left[ \frac{243}{16} + \frac{323 + 109m}{6 - 4m + m^2} - 96 \frac{23 - 11m}{[6 - 4m + m^2]^2} - \frac{215 - 53m}{30 - 4m + m^2} \right] m^4,$$

$$\frac{a_2 - a_{-2}}{a_0} = \frac{3}{32} \left[ \frac{243}{8} + \frac{175 + 563m}{6 - 4m + m^2} - 48 \frac{89 - 32m}{[6 - 4m + m^2]^2} + 5 \frac{361 - 10m}{30 - 4m + m^2} \right] m^4.$$

The evident objection to this form for the coefficients is that it makes the several terms very large, and of signs such that they nearly neutralize each other, the sum being very much smaller than any of the component terms. However it may be possible to remedy this imperfection by admitting three terms into the numerators, but, in this way, the problem is indeterminate, infinite variety being possible.

It is remarkable that none of our system of divisors can vanish for any real value of  $m$ , since the quadratic equations, obtained by equating them to zero, have all imaginary roots. In this they differ from the binomial divisors met with when the integration is effected in approximations arranged according to ascending powers of the disturbing force.

It is well known that the infinite series, obtained from the development, in ascending powers of  $m$ , of any fraction whose numerator is an entire function of  $m$ , and its denominator any integral power of a divisor of the previously mentioned series, is convergent, provided that  $m$  lies between the two square roots of the absolute term of the divisor. Hence any finite expression in  $m$ , involving these divisors, can be developed in such a series, provided that the numerical value of this parameter is less than  $\sqrt{6}$ . The same, however, cannot be asserted when the expression really forms an infinite series, as it is in the equation just given for the value of  $\frac{a_i}{a_0}$ . Yet, on account of the simplicity with which these quantities can be expressed in this form,  $a_1$  and  $a_{-1}$  containing each a single term, with an error of the sixth order only, this limit is worthy of attention.

If the parameter  $m$ , hitherto employed by the lunar theorists, is taken as the quantity in powers of which to expand the value of  $a_i$ , we shall have

$m = \frac{m}{1-m}$ . And, substituting this value, the principal divisor  $6 - 4m + m^2$  becomes  $6 - 16m + 11m^2$ . Thus the limits, between which  $m$  must be contained, in order that convergent series may be obtained where this divisor intervenes, are  $\pm \sqrt{\frac{6}{11}}$ . When we consider how little, in the case of our moon,  $m$  exceeds  $m$ , it will be plain that the series, in terms of  $m$ , are likely to be much more convergent than those in terms of  $m$ .

If we inquire what function of  $m$ , of the form  $\frac{m}{1+\alpha m}$ , the quantity

$$\frac{M}{[6 - 4m + m^2]^k}$$

can be expanded in powers of, with the greatest convergency, it is easily found that  $\alpha$  should be  $-\frac{1}{3}$ . Then putting

$$m = \frac{\mathfrak{m}}{1 + \frac{1}{3}\mathfrak{m}},$$

the divisor  $6 - 4m + m^2$  is changed into

$$6 + \frac{1}{3}\mathfrak{m}^2,$$

and there is introduced the additional divisor  $1 + \frac{1}{3}\mathfrak{m}$ . Here the series will be convergent provided  $\mathfrak{m}$  is less than 3. It is true the terms involving the succeeding divisors  $30 - 4m + m^2$ , &c., are not benefited by this change of parameter, but as they play an inferior rôle in this matter, I have chosen  $\mathfrak{m}$  as the parameter for the developments of the coefficients  $a_i$  in series of ascending powers.

To illustrate this matter, we have, in terms of the parameter  $\mathfrak{m}$ , and with errors of the sixth order,

$$\frac{a_1 + a_{-1}}{a_0} = - \left[ \frac{2 + \frac{1}{6}\mathfrak{m}}{1 + \frac{1}{18}\mathfrak{m}^2} - \frac{1}{1 + \frac{1}{3}\mathfrak{m}} \right] \mathfrak{m}^2,$$

$$\frac{a_1 - a_{-1}}{a_0} = \left[ \frac{5 + \frac{7}{6}\mathfrak{m}}{1 + \frac{1}{18}\mathfrak{m}^2} - \frac{7}{1 + \frac{1}{3}\mathfrak{m}} + \frac{\frac{27}{8}}{[1 + \frac{1}{3}\mathfrak{m}]^2} \right] \mathfrak{m}^2.$$

Expanding these expressions in powers of  $m$ , we get

$$\frac{a_1 + a_{-1}}{a_0} = - \left[ m^2 + \frac{1}{2} m^3 - \frac{2}{9} m^4 + \frac{1}{36} m^5 + \dots \right],$$

$$\frac{a_1 - a_{-1}}{a_0} = \frac{11}{8} m^2 + \frac{5}{4} m^3 + \frac{5}{72} m^4 - \frac{11}{36} m^5 + \dots$$

Let these series be compared with those which correspond to them in the lunar theories of Plana or Delaunay, viz:

$$m^2 + \frac{19}{6} m^3 + \frac{131}{18} m^4 + \frac{383}{27} m^5 + \dots,$$

$$\frac{11}{8} m^2 + \frac{59}{12} m^3 + \frac{893}{72} m^4 + \frac{2855}{108} m^5 + \dots$$

The superiority of the former, in convergence and simplicity of numerical coefficients, is manifest.

Much more might be said relative to possible modes of developing the coefficients  $a_i$  in series, but we content ourselves with giving their values expanded in powers of  $m$ , the series being carried to terms of the ninth order inclusive. The denominators of the numerical fractions are written as products of their prime factors, as, in this form, they can be more readily used, the principal labor in performing operations on these series being the reduction of the several fractional coefficients, to be added together, to a common denominator.

$$\frac{a_1}{a_0} = \frac{3}{2^4} m^2 + \frac{1}{2} m^3 + \frac{7}{2^2 \cdot 3} m^4 + \frac{11}{2^2 \cdot 3^2} m^5 - \frac{30749}{2^{12} \cdot 3^3} m^6 - \frac{1010521}{2^{11} \cdot 3^4 \cdot 5} m^7 - \frac{18445871}{2^{10} \cdot 3^5 \cdot 5^2} m^8$$

$$- \frac{2114557853}{2^{12} \cdot 3^6 \cdot 5^3} m^9 \dots$$

$$\frac{a_{-1}}{a_0} = - \frac{19}{2^4} m^2 - \frac{5}{3} m^3 - \frac{43}{2^2 \cdot 3^2} m^4 - \frac{14}{3^3} m^5 - \frac{7381}{2^{10} \cdot 3^4} m^6 + \frac{8574153}{2^{11} \cdot 3^5 \cdot 5} m^7 + \frac{55218889}{2^9 \cdot 3^6 \cdot 5^2} m^8$$

$$+ \frac{13620153029}{2^{12} \cdot 3^7 \cdot 5^3} m^9 \dots$$

$$\frac{a_2}{a_0} = \frac{25}{2^8} m^4 + \frac{803}{2^7 \cdot 3 \cdot 5} m^5 + \frac{6109}{2^5 \cdot 3^2 \cdot 5^2} m^6 + \frac{897599}{2^8 \cdot 3^3 \cdot 5^3} m^7 + \frac{237203647}{2^{16} \cdot 3^2 \cdot 5^4} m^8 - \frac{44461407673}{2^{15} \cdot 3^4 \cdot 5^5 \cdot 7} m^9 \dots$$

$$\frac{a_{-2}}{a_0} = 0 m^4 + \frac{23}{2^7 \cdot 5} m^5 + \frac{299}{2^5 \cdot 3 \cdot 5^2} m^6 + \frac{56339}{2^8 \cdot 3^2 \cdot 5^3} m^7 + \frac{79400351}{2^{16} \cdot 3^2 \cdot 5^4} m^8 + \frac{8085846833}{2^{14} \cdot 3^4 \cdot 5^5 \cdot 7} m^9 \dots$$

$$\frac{a_3}{a_0} = \frac{883}{2^{12} \cdot 3} m^6 + \frac{27943}{2^{11} \cdot 5 \cdot 7} m^7 + \frac{12275527}{2^{10} \cdot 3^2 \cdot 5^2 \cdot 7^2} m^8 + \frac{27409853579}{2^{12} \cdot 3^4 \cdot 5^3 \cdot 7^3} m^9 \dots$$

$$\frac{a_{-3}}{a_0} = \frac{1}{2^6 \cdot 3} m^6 + \frac{71}{2^7 \cdot 3 \cdot 5} m^7 + \frac{46951}{2^8 \cdot 3^2 \cdot 5^2 \cdot 7} m^8 + \frac{14086643}{2^7 \cdot 3^4 \cdot 5^3 \cdot 7^2} m^9 \dots$$

$$\frac{a_4}{a_0} = \frac{8537}{2^{16}} m^8 + \frac{111809667}{2^{17} \cdot 3^2 \cdot 5 \cdot 7^2} m^9 \dots$$

$$\frac{a_{-4}}{a_0} = \frac{28}{2^{11} \cdot 3} m^8 + \frac{1576553}{2^{17} \cdot 3^2 \cdot 7^2} m^9 \dots$$

These values being substituted in the equations

$$\begin{aligned} r \cos v &= \sum_i a_i \cos 2i\tau, \\ r \sin v &= \sum_i a_i \sin 2i\tau, \end{aligned}$$

and the parameter changed to  $m$ , we get

$$\begin{aligned} r \cos v &= a_0 \left\{ 1 + \left[ -m^2 - \frac{1}{2} m^3 + \frac{2}{9} m^4 - \frac{1}{36} m^5 - \frac{106411}{331776} m^6 + \frac{427339}{497664} m^7 \right. \right. \\ &\quad \left. \left. + \frac{25289037}{14929920} m^8 - \frac{732931}{37324800} m^9 \dots \right] \cos 2\tau \right. \\ &\quad \left. + \left[ \frac{25}{256} m^4 + \frac{311}{960} m^5 + \frac{9849}{28800} m^6 - \frac{5831}{216000} m^7 \right. \right. \\ &\quad \left. \left. - \frac{164645363}{552960000} m^8 - \frac{11321875589}{19353600000} m^9 \dots \right] \cos 4\tau \right. \\ &\quad \left. + \left[ \frac{299}{4096} m^6 + \frac{30193}{107520} m^7 + \frac{379549}{1003520} m^8 + \frac{181908179}{1580544000} m^9 \dots \right] \cos 6\tau \right. \\ &\quad \left. + \left[ \frac{11347}{196608} m^8 + \frac{2850381}{9031680} m^9 \dots \right] \cos 8\tau + \dots \right\}, \\ r \sin v &= a_0 \left\{ \left[ \frac{11}{8} m^2 + \frac{5}{4} m^3 + \frac{5}{72} m^4 - \frac{11}{36} m^5 - \frac{101123}{331776} m^6 - \frac{512239}{276480} m^7 \right. \right. \\ &\quad \left. \left. - \frac{269023019}{74649600} m^8 - \frac{151872119}{93312000} m^9 \dots \right] \sin 2\tau \right. \\ &\quad \left. + \left[ \frac{25}{256} m^4 + \frac{121}{480} m^5 + \frac{5623}{28800} m^6 - \frac{17149}{482000} m^7 \right. \right. \\ &\quad \left. \left. - \frac{3500287}{11520000} m^8 - \frac{43885512859}{58060800000} m^9 \dots \right] \sin 4\tau \right. \\ &\quad \left. + \left[ \frac{769}{12288} m^6 + \frac{24481}{107520} m^7 + \frac{4419847}{15052800} m^8 + \frac{398814169}{4741632000} m^9 \dots \right] \sin 6\tau \right. \\ &\quad \left. + \left[ \frac{9875}{196608} m^8 + \frac{32608451}{144506880} m^9 \dots \right] \sin 8\tau + \dots \right\}. \end{aligned}$$

Our final differential equations are capable of furnishing only the ratios of the coefficients  $a_i$ , hence we must have recourse to one of the original

equations if we wish to determine  $a_0$  as a function of  $n$  and  $\mu$ . By substituting the values

$$u = \sum_i a_i \zeta^{2i+1}, \quad s = \sum_i a_{-i-1} \zeta^{2i+1},$$

in the differential equation

$$\left[ D^2 + 2mD + \frac{3}{2} m^2 - \frac{\chi}{(us)^{\frac{3}{2}}} \right] u + \frac{3}{2} m^2 s = 0,$$

we obtain

$$\frac{\chi u}{(us)^{\frac{3}{2}}} = \sum_i \left\{ \left[ (2i+1+m)^2 + \frac{1}{2} m^2 \right] a_i + \frac{3}{2} m^2 a_{-i-1} \right\} \zeta^{2i+1}.$$

Considering only the term of this, for which  $i = 0$ , and supposing that the coefficient of  $\zeta$  in the expansion of  $\frac{a_0^2 u}{(us)^{\frac{3}{2}}}$  is denoted by  $J$ , we shall have

$$\frac{\chi}{a_0^3} J = 1 + 2m + \frac{3}{2} m^2 + \frac{3}{2} m^2 \frac{a_{-1}}{a_0}.$$

For brevity call the right member of this  $H$ ; then, since

$$\chi = \frac{\mu}{(n-n')^2} = \frac{\mu}{n^2} (1+m)^2,$$

we shall have

$$a_0 = \left[ \frac{\mu}{n^2} \right]^{\frac{1}{3}} \left[ \frac{J(1+m)^2}{H} \right]^{\frac{1}{3}}.$$

The value of  $H$  is readily obtained from the value of  $\frac{a_{-1}}{a_0}$  given above, and  $J$  must be found by substituting the values

$$u = \sum_i a_i \zeta^{2i+1}, \quad s = \sum_i a_{-i-1} \zeta^{2i+1}$$

in  $\frac{a_0^2 u}{(us)^{\frac{3}{2}}}$ , and taking the coefficient of  $\zeta$ . We get

$$\begin{aligned} J = 1 &+ \left[ \frac{a_1 + a_{-1}}{a_0} \right]^2 \left[ \frac{3}{4} + \frac{45}{64} \left[ \frac{a_1 + a_{-1}}{a_0} \right]^2 + \frac{15}{8} \frac{a_1 a_{-1}}{a_0} - \frac{15}{2} \frac{a_2 + a_{-2}}{a_0} \right] \\ &+ \frac{a_2 + a_{-2}}{a_0} \left[ \frac{3}{4} \frac{a_2 + a_{-2}}{a_0} + 6 \frac{a_1 a_{-1}}{a_0^2} \right] + 6 \frac{a_1 + a_{-1}}{a_0} \frac{a_1 a_2 + a_{-1} a_{-2}}{a_0^2} \\ &+ 3 \frac{a_1 a_{-1}}{a_0^2} + 45 \frac{a_1^2 a_{-1}^2}{a_0^4} + 3 \frac{a_2 a_{-2}}{a_0^2}, \end{aligned}$$

where the terms neglected are, at lowest, of the tenth order with respect to  $m$ . And, explicitly in terms of this parameter,

$$J = 1 + \frac{21}{2^8} m^4 - \frac{31}{2^5} m^5 - \frac{53}{2^4} m^6 - \frac{2707}{2^6 \cdot 3^2} m^7 - \frac{4201213}{2^{16} \cdot 3^3} m^8 + \frac{14374939}{2^{15} \cdot 3^3 \cdot 5} m^9 \dots$$

By means of which there is obtained

$$a_0 = \left[ \frac{\mu}{n^2} \right]^{\frac{1}{3}} \left[ 1 - \frac{1}{6} m^2 + \frac{1}{3} m^3 + \frac{407}{2304} m^4 - \frac{67}{288} m^5 - \frac{45293}{41472} m^6 \right. \\ \left. - \frac{8761}{6912} m^7 - \frac{4967441}{7962624} m^8 + \frac{14829273}{39813120} m^9 \dots \right],$$

or, in terms of the parameter  $m$ ,

$$a_0 = \left[ \frac{\mu}{n^2} \right]^{\frac{1}{3}} \left[ 1 - \frac{1}{6} m^2 + \frac{4}{9} m^3 - \frac{163}{768} m^4 - \frac{1147}{5184} m^5 - \frac{79859}{124416} m^6 \right. \\ \left. + \frac{4811}{10368} m^7 + \frac{9520295}{71663616} m^8 + \frac{189240651}{1074954240} m^9 \dots \right].$$

The quantity  $\left[ \frac{\mu}{n^2} \right]^{\frac{1}{3}}$  is usually designated  $\alpha$  by the lunar-theorists; and,

to make this appear as a factor of the expressions for  $r \cos v$  and  $r \sin v$ , it would be necessary to multiply all the coefficients by the second factor of the preceding expression for  $a_0$ . It seems simpler however to retain  $a_0$  as the factor of linear magnitude; for the astronomers have preferred to derive the constant of lunar parallax from direct observation of the moon, or, in other words, they have preferred to consider  $\mu$  as a seventh element of the orbit; with this view of the matter, there is no incongruity in making  $a_0$  everywhere replace  $\mu$ .

The expression for  $a_0$  can be obtained in several other ways, which lead to more symmetrical formulæ, and which also serve for verification of all the preceding developments. If, in the preceding equation giving the value of  $\frac{xu}{(us)^{\frac{3}{2}}}$  in terms of  $\zeta$ , we attribute to  $\tau$  the value 0, or, which is equivalent, make  $\zeta = 1$ , we shall have  $u = s = \sum_i a_i$ , and, consequently

$$\frac{x}{[\sum_i a_i]^2} = \sum_i [(2i + 1 + m)^2 + 2m^2] a_i.$$

And thus, mindful of the value of  $x$  given above, we get

$$a_0 = \left[ \frac{\mu}{n^2} \right]^{\frac{1}{3}} \left[ \frac{(1 + m)^2}{\sum_i [(2i + 1 + m)^2 + 2m^2] \frac{a_i}{a_0} \cdot \left[ \sum_i \frac{a_i}{a_0} \right]^2} \right]^{\frac{1}{3}}.$$

Again the differential equation

$$\frac{d^2y}{d\tau^2} + 2m \frac{dx}{d\tau} + \frac{x}{r^3} y = 0$$

gives

$$\frac{\kappa}{r^3} \cdot y = \Sigma_i \cdot [(2i+1+m)^2 - m^2] a_i \sin (2i+1) \tau,$$

and, attributing to  $\tau$  the special value  $\frac{\pi}{2}$ ,

$$\frac{\kappa}{[\Sigma_i \cdot (-1)^i a_i]^2} = \Sigma_i \cdot (-1)^i (2i+1) (2i+1+m) a_i.$$

Whence

$$a_0 = \left[ \frac{\mu}{n^2} \right]^{\frac{1}{3}} \left[ \frac{(1+m)^2}{\Sigma_i \cdot (-1)^i (2i+1) (2i+1+m) \frac{a_i}{a_0} \cdot \left[ \Sigma_i \cdot (-1)^i \frac{a_i}{a_0} \right]^2} \right]^{\frac{1}{3}}.$$

When  $j = 0$  in the first equation of condition for determining the coefficients  $a_i$ , we get a formula expressing  $C$  in terms of these quantities, viz.,

$$C = \Sigma_i \cdot \left[ (2i+1+2m)^2 + \frac{1}{2} m^2 \right] a_i^2 + \frac{9}{2} m^2 \Sigma_i \cdot a_i a_{-i-1},$$

or neglecting terms of the eighth and higher orders,

$$\begin{aligned} C &= a_0^2 \left[ 1 + 4m + \frac{9}{2} m^2 + (9 + 12m + \frac{9}{2} m^2) \frac{a_1^2}{a_0^2} + (1 - 4m + \frac{9}{2} m^2) \frac{a_{-1}^2}{a_0^2} + 9m^2 \frac{a_{-1}}{a_0} \right] \\ &= a_0^2 \left[ 1 + 4m + \frac{9}{2} m^2 - \frac{1147}{2^7} m^4 - \frac{1899}{2^6 \cdot 3} m^5 - \frac{2047}{2^8} m^6 + \frac{3737}{2^4 \cdot 3^3} m^7 \right]. \end{aligned}$$

But the  $C$  of Chap. I is obtained by multiplying this  $C$  by  $\frac{1}{2} \nu^2 = \frac{1}{2} \frac{n^2}{(1+m)^2}$ .  
Hence, substituting for  $a_0$  its value, we have

$$C = \frac{1}{2} (\mu n)^{\frac{2}{3}} \left[ 1 + 2m - \frac{5}{6} m^2 - m^3 - \frac{1319}{288} m^4 - \frac{67}{144} m^5 - \frac{2879}{1296} m^6 - \frac{1321}{1296} m^7 \right],$$

as there stated.

We propose now to reduce the preceding formulæ to numerical results. For this purpose we assume

$$n = 17325594''.06085,$$

$$n' = 1295977''.41516,$$

which give

$$m = \frac{n'}{n - n'} = 0.08084\ 89338\ 08312,$$

$$m^2 = 0.00653\ 65500\ 97941,$$

$$m^3 = 0.00052\ 84731\ 06203,$$

$$m^4 = 0.00004\ 27264\ 87183,$$

$$m = 0.08308\ 81293\ 65.$$

The numerical value of  $\mu$  being substituted in the series, we obtain

$$a_0 = 0.99909\ 31419\ 62 \left[ \frac{\mu}{n^2} \right]^{\frac{1}{3}},$$

$$\begin{aligned} r \cos v = a_0 & [ 1 - 0.00718\ 00394\ 55 \cos 2\tau \\ & + 0.00000\ 60424\ 59 \cos 4\tau \\ & + 0.00000\ 00325\ 76 \cos 6\tau \\ & + 0.00000\ 00001\ 80 \cos 8\tau ], \end{aligned}$$

$$\begin{aligned} r \sin v = a_0 & [ 0.01021\ 14543\ 96 \sin 2\tau \\ & + 0.00000\ 57148\ 79 \sin 4\tau \\ & + 0.00000\ 00274\ 99 \sin 6\tau \\ & + 0.00000\ 00001\ 57 \sin 8\tau ]. \end{aligned}$$

(To be Continued.)

